Mathematical Misconceptions Commonly Exhibited by Entering Tertiary Mathematics Students

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In terms of subsequent learning of mathematics, the importance of mastering a limited number of basic concepts cannot be over-estimated. This paper examines the nature and frequency of mathematical misconceptions which have been commonly exhibited in tests at two tertiary institutions. The frequency of mathematical misconceptions seen is of great concern. Hopefully information about which misconceptions occur commonly will lead mathematics educators to place greater emphasis on the teaching of these concepts, thereby causing a decrease in the frequency of the related misconceptions.

Introduction

Due to the sequential nature of many branches of mathematics, it is extremely important that individuals attempting to learn mathematics at a particular level first master the fundamental mathematics which is pre-requisite knowledge for success at that higher level. The mastery of this pre-requisite knowledge depends very heavily on individuals having a thorough understanding of a limited number of basic concepts. As the learning and understanding of these concepts as well as the ability to use this knowledge is so crucial to subsequent learning of mathematics, it is of great interest to mathematics educators (teachers, lecturers and tutors) to be able to ascertain information about the types of misconceptions commonly held by students and the frequency with which these misconceptions occur.

This paper will consider these mathematical misconceptions from the perspective of mathematics students making the transition from secondary mathematics to tertiary mathematics. It is argued that the understanding of the basic mathematical concepts which should be acquired in secondary school is crucial to the preparedness of mathematics students to undertake a tertiary course involving some mathematics units. The preparedness of mathematics students is important from both the viewpoints of prospective tertiary mathematics students and that of the Victorian Government which had a stated policy that it was committed to increasing participation in post-secondary mathematics education (Victorian Government 1987, p. 89) and that educational programs from Years P--12 should provide all students with a sound preparation for further schooling (Ministry of Education (Victoria) -- Ministerial Paper Number 6 1984, pp. 6.11).

A National Statement on Mathematics for Australian Schools (Australian Education Council 1990) hereafter referred to as the National Statement, also recognises the importance of this issue. One of the goals for mathematics in Australian schools included in the National Statement was that 'as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively' (p. 18).

This goal is particularly significant when one considers that 'contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies' (Blyth and Calegari 1985, p. 312). Also, 'statistics collected by the MAV (Mathematical Association of Victoria) show that 75% of all tertiary courses require a pass in HSC mathematics' (Blyth and Calegari 1985, p. 312).

A great deal has been written on the subject of mathematical misconceptions. There are many types of mathematical misconceptions, 'and a complete list may not even be practical' (Davis 1984, p. 335). A number of authors have written about how a consideration of pupils' misconceptions in mathematics can be used to develop teaching techniques which are directed towards diagnosing and eliminating these misconceptions (Bell 1982, Farrell 1992, Margulies 1993, Perso 1992).

Papers such as this one have the potential to assist in the direction of the teaching of mathematics in such a way as to help to correct some of these misconceptions, or at least to decrease the frequency of their occurrence. This desirable outcome may come about by making mathematics teachers aware of some of the more common misconceptions and how frequently they occur. By providing mathematics educators with information about which misconceptions are common and how frequently they occur, it is hoped that this may lead to a greater emphasis being placed on the teaching of areas in which misconceptions occur frequently, and therefore provide a better mathematics education for students.

In this paper, the mathematical misconceptions have been drawn from a Diagnostic Test used at the University of Melbourne, a first year examination at the University of Melbourne, and a test given at LaTrobe University (LaTrobe). A list of these misconceptions and their frequencies of occurence is provided, along with a discussion of why they may have occurred.

The 1992 Diagnostic Test

Since 1989 a Diagnostic Test has been used at the University of Melbourne to identify the level of preparedness of students to undertake a tertiary mathematics course and to pinpoint areas of weakness in the backgrounds of the students (Barrington and Carbone 1991). Results from the tests are used to advise each student which first year mathematics unit is most appropriate for that student and also to determine whether a student has a need for extra assistance.

While analysing the 1992 Diagnostic Test (1992 DT), a number of misconceptions were seen to have occurred with a relatively high frequency and a list of these was compiled. This research does not provide a list of all of the mathematical misconceptions which were exhibited, but considers a relatively small number of mathematical misconceptions with high frequency of occurrence (this was taken to be greater than 10% of those who attempted the question, or about 5% where a question already had a misconception with greater than 10% frequency recorded). A tally was kept of the number of students who attempted each question which had a high rate of misconceptions, how many gave the correct answer to these questions, how many exhibited the misconception, and how many gave another incorrect answer. A summary of these statistics is given in Table 1 below. The items referred to in Table 1 are listed and discussed in following paragraphs.

In Table 1, and in the discussion which follows below, the proportion of misconceptions refers to the frequency of that misconception with respect to the total number of students who attempted that question i.e. if 1000 students had sat a test, and, of these, 800 had attempted a particular question with 200 demonstrating a misconception, then that misconception will be said to have occurred in 25% of cases (not 20%). The reason for adopting this procedure is that it is not clear why a student did not attempt a particular question -- there is no evidence which suggests the presence or the absence of the misconception being considered.

Table 1 -- Frequencies/percentages of Misconceptions

Table 1 summarises the frequencies of the answers given by the 394 V.C.E. (Victorian Certificate of Education) mathematics subject 'Change and Approximation 3/4 Extensions' (C & A 3/4 X) students who sat the 1992 Diagnostic Test. '% Att.' refers to the percentage of students who had attempted that question, % 's in each of the other categories refer to the percentage of students in that category of those who had attempted the question. Numbers in brackets after question numbers indicate the first and second misconceptions.

Question	Attempted (% Att.)	Correct (% Correct)	Misconception (% Miscon.)	Other (% Other)
A3	332 (84.3)	148 (44.6)	91 (27.4)	93 (28.0)
A5	341 (86.6)	227 (66.6)	45 (13.2)	69 (20.2)
A7(1)	378 (95.9)	215 (56.9)	78 (20.6)	85 (22.5)
A7(2)			18 (4.8)	
A8(1)	363 (92.1)	215 (59.2)	48 (13.2)	100 (27.6)
A8(2)			20 (5.5)	
A9	283 (71.8)	127 (44.9)	70 (24.7)	86 (30.4)
B3	374 (94.9)	235 (62.8)	67 (17.9)	72 (19.3)
D4 i	229 (58.1)	20 (8.7)	82 (35.8)	127 (55.5)

The misconceptions and their frequencies of occurrence contained in this section have been drawn from the test booklets of the students who had completed C & A 3/4 X in 1991 and then attempted the 1992 DT. All questions on the test were 'short answer' and only the answer was marked.

It should be noted that not all of the so-called mathematical misconceptions shown here are, literally speaking, misconceptions. Some may simply be a result of misreading or misinterpreting the question. These mistakes are nevertheless important, the level of their importance being governed by the nature of the mistake and the frequency with which it occurs. The question numbers shown below are those used in the 1992 DT.

In each case below, the correct solution is shown immediately after the question. It should be noted that the method shown is the one which was anticipated, and other correct solutions may be possible.

A3. Factorise $(2x+y)^2 - x^2$.

$$\begin{aligned}(2x+y)^2 - x^2 &= ((2x+y)-x)((2x+y)+x)\\ &= (x+y)(3x+y).\end{aligned}$$

Of those who answered this question, 27% gave the answer $3x^2+4xy+y^2$. It is unlikely that all of them had the same misconception. Since many students had demonstrated (in other questions) that they knew the meaning of the word factorise, this answer may be due to some not recognising that a difference of squares was involved, some expanding the given expression and subsequently forgetting the purpose of the question, while others may have factorised the difference of squares and then expanded having forgotten the purpose of the question.

A5. Simplify $\log_{10}45+\log_{10}2-\log_{10}15$.

$$\begin{aligned}\log_{10}45+\log_{10}2-\log_{10}15 &= \log_{10}(45\times 2\div 15)\\ &= \log_{10}6.\end{aligned}$$

Some 13% of those who answered the question solved it in the following way :

$$\log_{10}45+\log_{10}2-\log_{10}15 = \log_{10}(45+2-15) = \log_{10}32.$$

The common misconception exhibited here is that $\log A + \log B = \log(A+B)$. This idea may be due to an overgeneralisation of the process used to find the sum of real numbers. It may also be that the students who have exhibited this misconception have overgeneralised linearity of operation regardless of the operator. Two other examples of this type of misconception are given below :

(i) $\sqrt{x^2+y^2} = x+y$ (where $x+y \geq 0$); (ii) $\sin(A+B) = \sin A + \sin B$.

A7. Find $\{x:(x-1)(x^2-3x)=0\}$.

$(x-1)(x^2-3x) = (x-1)x(x-3) = 0$; so x=0,1 or 3.

About 21% of those who answered the question missed the value x=0 and gave the answer x=1,3. Another common mistake (5%) was to miss the values x=0,3 and give the answer x=1. This appears to be due to the second bracket in the question being ignored or disregarded.

A8. $\{x:2x+4<5x+10\}$.

$-6<3x$; $x>-2$.

Many students (13%) appear to have written down the inequation $-6<3x$ and then incorrectly that $x<-2$. It is a matter of conjecture as to the proportions who were careless or who had misconceptions about the meanings of < and >. It is also possible that some students have tried to get x on the left side of the inequation by first writing $-6<3x$ and then dividing both sides by -3 without reversing the inequality. This error may have been due to a misconception or a lack of care. A further 6% of students gave the answer $x=-2$ to this question. This may have been due to being careless with < , > and = or to a misunderstanding of what these symbols mean or what is expected of them in this type of question. It may also be that this incorrect answer is a result of confusing the method of approach to this type of question. When sketching the graph of an inequality, students are often taught to sketch the equality first, and then consider which region is required. Some students may have adopted this approach to the given inequation, but, having solved the equation, have forgotten to complete the solution.

A9. Solve for x : $\frac{3}{x}-\frac{4}{a}=\frac{5}{b}$ where a and b are positive real numbers.

$$\frac{3}{x}=\frac{5}{b}+\frac{4}{a}$$

$$=\frac{5a+4b}{ab}$$

$$x=\frac{3ab}{5a+4b}$$

Of those who answered this question, 25% appear to have first written down $\frac{x}{3}-\frac{a}{4}=\frac{b}{5}$ (or similar). Apparently these students believed that combinations of fractions could be 'flipped over' without any consideration of a common denominator. This misconception may be a result of believing that you can do anything to an equation as long as you do the same thing to both sides of the equation. Here students have taken the reciprocal of each term rather than each side of the equation.

B3. Write down the derivative of $f(x): f(x) = 2x^3e^x$.

$$f'(x) = \frac{d}{dx}(2x^3) \times e^x + 2x^3 \times \frac{d}{dx}(e^x)$$
$$= 6x^2e^x + 2x^3e^x.$$

Some 18% of those who answered the question wrote down the answer incorrectly as $f'(x) = 6x^2e^x$. This may be a case of many students forgetting the product rule for differentiation. Alternatively, some of the students may have treated e^x as a constant while others might have thought that the derivative of a product was the product of the derivatives i.e. if $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h'(x)$.

D4i. Given that $\sin A = \frac{3}{4}$ and that $\frac{\pi}{2} < A < \pi$, evaluate the following : $\cos A$.

$\cos^2 A = 1 - \sin^2 A = 1 - \frac{9}{16} = \frac{7}{16}$; $\cos A = \pm\frac{\sqrt{7}}{4}$ but we are given that A is in the second quadrant, so $\cos A = -\frac{\sqrt{7}}{4}$.

The answer $\cos A = \frac{\sqrt{7}}{4}$ was given by 36% of those who attempted this question. This appears to contain the misconception that $x^2 = C^2$ has only one root : $x = C$; the negative root is missed. It is also likely that some students either have not read the question carefully enough or have disregarded the fact that A is in the second quadrant.

A Cautionary Note

When considering the frequency of mathematical misconceptions exhibited by students attempting the 1992 DT, it is important to note that they are some of the better students in the state of Victoria (according to their Tertiary Entrance Scores), who had succeeded in what was considered to be the most demanding Victorian Certificate of Education (V.C.E.) mathematics course, that being C & A 3/4 X. The Tertiary Entrance Score or Anderson score as it was commonly known, was calculated for each student by adding together the student's marks for their four best subjects and adding to this ten per cent of the marks for the other subjects attempted for which the student had scored at least 40%. The students being considered here had an average Anderson score of about 328 which corresponds to the thirteenth percentile of all students who completed the V.C.E. in 1991, and they had an average C & A 3/4 X mark of about 76%. It is worth considering that if the misconceptions described in this paper occur with such frequency amongst the better students in the state, with what frequency could we expect them to occur amongst students of lesser standing?

It should however be acknowledged that as the 1992 DT was held at the end of February, some of the misconceptions exhibited may be due to the length of the break from studying mathematics.

Another Important Misconception

The misconception discussed in this section was exhibited in the first semester 1992 Mathematics 100 Examination at the University of Melbourne. The question asked on the examination required students to find the area between two cubic curves. To find the points of intersection of these curves, students equated them to give the equation $x^3 - 3x + 2 = 0$. Students were given that one of the solutions was $x=1$.

$(x-1)(x^2 + x - 2) = 0$

(the step above could be done by inspection or by long division)

$(x-1)(x-1)(x-2) = 0$

$(x-1)^2(x-2) = 0$

so $x=-2$ or 1 .

The markers, which included the author, discussed the performance of students subsequent to the completion of marking. One misconception was highlighted by each of the markers. This had to do with the incorrect use of what is sometimes referred to as the null factor law -- if the product of two (or more) numbers is zero, then at least one of those numbers must be equal to zero, i.e. if $x(x-3)=0$, then $x=0$ or $x-3=0$; and so $x=0$ or 3. This law appears to have been modified to yield the following : if the product of two (or more) numbers is equal to a number, then at least one of those numbers must be equal to that number, i.e. if $x(x-3)=1$, then $x=1$ or $x-3=1$; and so $x=1$ or 4. The equation was solved incorrectly by many of the students in the following way :

$$x^3-3x+2=0$$

$$x^3-3x=-2$$

$$x(x^2-3)=-2$$

so $x=-2$ or $x^2-3=-2$

$$x^2=1$$

$x=1$. Some of the students did get the two solutions $x=\pm1$ but many demonstrated the same misconception as that shown previously in D4i.

The markers discussed the frequency with which this misconception occurred and each stated that the frequency was somewhere between 20 and 25% of students. The nature of this misconception shocked the markers, but not nearly as much as the frequency of its occurrence. No exact count was taken of the frequency of occurrence of this misconception, but all markers reported a similar estimate of the frequency quite independently of one another. It is nevertheless quite likely that there could be a considerable error in these estimates. Even if this is so, the fact that the misconception occurred with such frequency so as to be noticed by the markers as a common problem amongst a large number of the students, and to cause such a widespread level of concern amongst the markers, means that it is important. No other misconceptions caused widespread concern or discussion.

It was surprising that this misconception had persisted to this stage especially with this group of 'well above average' students (as seen by their tertiary entrance scores). The null factor law is not taught in Mathematics 100 as students are expected to have this knowledge prior to entry in this unit.

It is ironic to note that students who exhibited both of the misconceptions shown above, had the correct two solutions. In other words, those students who overgeneralised the null factor law when the right hand side of the equation was equal to -2, and then put 1 instead of ±1 when solving $x^2=1$, obtained the two solutions x=-2 or 1, and these were the only two correct solutions.

Some Other Misconceptions

At a similar time to the 1992 DT, students at LaTrobe University were required to sit a mathematics test. As these tests were administered at approximately the same time, it is of interest to consider the misconceptions exhibited on the LaTrobe test. The misconceptions below are drawn from an analysis of a test given to students studying mathematics at LaTrobe. No information is provided which indicates which V.C.E. mathematics courses had been attempted in the previous year, or what levels had been achieved. However, the misconceptions exhibited can be considered to be important as the test was comprised of 'very simple questions -- no question was above early year 11 standard' (Worley 1993, p. 1). In this analysis it is stated that 'the lack of preparation of our students in pre-calculus skills has become a problem of major concern for us at LaTrobe' (Worley 1993, p. 1). It is interesting to note that many of the misconceptions shown below appear on the list of misconceptions provided in 'Algebraic Atrocities' (Margulies 1993, p. 41). Examples from the analysis of the test are shown below :

Question : Given that $-\cos\frac{\pi}{6} = -\frac{7\sqrt{3}}{2}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, evaluate $\cos\frac{7\pi}{6}$. Students were expected to know that $\frac{7\pi}{6}$ is in the third quadrant, and the correct answer is $\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$. The most common response was $\cos\frac{7\pi}{6} = \frac{7\sqrt{3}}{2}$. This misconception is similar to that in Question A5 of the 1992 DT -- students have overgeneralised linearity of operation regardless of the operator.

Question : Simplify $\frac{x^2-5x-6}{x^2-2x-3}$.

$$\frac{x^2-5x-6}{x^2-2x-3} = \frac{(x-6)(x+1)}{(x-3)(x+1)} = \frac{(x-6)}{(x-3)}, \; x \neq -1 .$$

There were a wide variety of answers given, two of which are shown below :

$$\frac{\cancel{x}^2-5x-\cancel{6}}{\cancel{x}^2-2x-\cancel{3}} = \frac{-5x-2}{-2x-1};$$

$$\frac{\cancel{x}^2-5x-\cancel{6}}{\cancel{x}^2-2x-\cancel{3}} = -3x-3 .$$

In both cases, 'cancelling factors is clearly not understood by the majority of students. Clearly most have developed a general rule that any term on the numerator can be cancelled with any term on the denominator' (Worley 1993, p. 2). In the second case, it appears that the rule about subtracting indices when dividing numbers has been overgeneralised.

Question : Simplify $2^{-1}+3^{-1}$.

$$2^{-1}+3^{-1} = \frac{1}{2}+\frac{1}{3} = \frac{2+3}{6} = \frac{5}{6} .$$

About 24% of students put either 5^{-1} or 5^{-2}, to this question on indices showing 'a total lack of understanding of the rules' (Worley 1993, p. 2).

Question : Simplify 2^x+2^x . Either 2.2^x or 2^{x+1} was accepted as a correct answer to this question. However, '$2^x+2^x=4^x$ or 2^{2x} were overwhelmingly the most common errors, though 2^{x^2} also featured' (Worley 1993, p. 2). The same lack of understanding of index laws is evident here.

An approach to the problem

The vast majority of the misconceptions reported in this paper appear to be due to what Dina Tirosh (1990) refers to as external inconsistencies which she says 'occurs when a student's existing mathematical concept is incompatible with newly presented information' (p. 122). Insufficient understanding is likely to cause an inappropriate transfer of theorems which apply to one mathematical subculture into another in which they are invalid. As a result, students have over-generalised incorrect or inadequate rules. This categorisation is likely to be the major cause for most of the misconceptions exhibited in A5, A9 and B3 on the 92DT, the question taken from the Mathematics 100 exam, and in each of the questions from the LaTrobe test.

An approach which has been successful in helping many students overcome misconceptions is the 'conflict teaching approach' based on Piaget's notion of cognitive conflict. In this approach, teachers discuss with the learners the inconsistencies in the thinking of the learners in order to have the learners realise that their conceptions were inadequate and in need of modification (Tirosh 1990). Shlomo Vinner (1990) supports this approach and states that 'there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way' (p. 97).

Summary

A number of mathematical misconceptions have been exhibited (sometimes on very simple material) in tests at two different of tertiary institutions. The frequencies of these misconceptions are a matter of great concern for tertiary mathematics educators (lecturers and tutors).

Most of the basic concepts referred to in this paper are not taught in most of the tertiary mathematics units considered herein. Students are expected to have learned these concepts prior to entry to the respective unit. As a result, students who have not mastered those basic concepts can be severely disadvantaged in those tertiary mathematics units.

Many of the concepts shown previously *should* have been mastered long before a student gets to university. The fact that that is not the case with a large number of students (and a large number of the better students in the State) provides a very strong argument that the school system is not satisfying the needs of these students. These needs ought to be able to be satisfied and these concepts taught more effectively. The problem yet to be solved, is *how*?

It is the belief of the author that an effective method of minimising the frequency of mathematical misconceptions is to teach with emphasis placed on the understanding of a 'rule' rather than just remembering that rule. If misconceptions do occur (as they will) despite the best efforts of the teacher, the approach outlined above is a good starting point from which to begin to remedy the problem.

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